The Cycle Contraction Mapping Theorem

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1. Introduction

In the world of denotational semantics there are two principle approaches to defining fixed point semantics. Firstly there is the more usual *Tarski School* which uses a least fixed point theorem over domains constructed as complete partial orders. Less well known but still significant is the *Banach School* which uses the Banach contraction mapping theorem over domains constructed as complete metric spaces. As any Scott topology for a partial order has to be non Hausdorff and as every metric space has to be Hausdorff it is understandable that these two schools have had little in common to talk about. In particular the Banach school can be accused of denying the importance of a partial order solely on the grounds that there was no obvious way of defining one in a complete metric space.

Bridges can however be built between the Tarski and Banach schools. The Lawson [La87] approach is to construct a refined metric topology for each given Scott partial order topology. Being Hausdorff Lawson topologies are too fine for computational purposes, and so this approach side steps the important question of whether or not there are suitable distance functions to describe Scott topologies. Smyth [Sm87] has promoted the use of non-Hausdorff generalised metric spaces which include many Scott topologies. Smyth uses the quasi metric which is a non symmetric distance function having a natural definition of partial order. Although quasi metric topologies offer considerable promise as a means of unifying ideas from both the Tarski and Banach schools it is unlikely that the quasi metric itself is the most appropriate generalised metric for describing such topologies.

The *Cycle Contraction Mapping Theorem* is an extension of Banach's contraction mapping theorem for complete metric spaces to a class of quasi metric spaces. This theorem is formulated in terms of the author's *Partial Metric* [Ma92], a symmetric generalised metric with a quasi metric topology. Used for program correctness proofs such as absence of deadlock in Kahn Networks [Ka74] the cycle contraction mapping theorem cannot be formulated in terms of a quasi metric. Such proofs are novel in that they contain no reference to either partial objects or operational semantics.

2. Background Definitions and Results

Definition 2.1

A ***Metric*** [Su75] is a function $d : U^2 \rightarrow \Re$ such that ,

$$
\begin{array}{lll}
(\mathrm{M1}) & \forall\, x,y \in U \; . & x = y \;\Leftrightarrow\; d(x,y) = 0 \\
(\mathrm{M2}) & \forall\, x,y \in U \; . & d(x,y) = d(y,x) \\
(\mathrm{M3}) & \forall\, x,y,z \in U \; . & d(x,z) \leq d(x,y) + d(y,z)
\end{array}
$$

Definition 2.2

For each metric $d : U^2 \rightarrow \Re$ and $X \in {}^{\omega}U$, X is ***Cauchy*** if ,

$$\forall\; \varepsilon > 0 \;\; \exists\, k \in \omega \;\; \forall\, n,m > k \; . \; d(\, X_n \, , \, X_m \,) < \varepsilon$$

Definition 2.3

A metric is ***Complete*** if every Cauchy sequence converges.

The Banach Contraction Mapping Theorem

For each complete metric $d : U^2 \rightarrow \Re$ and function $f : U \rightarrow U$, f has a unique fixed point if ,

$$\exists\; 0 \leq c < 1 \;\; \forall\, x,y \in U \; . \; d(\, f(x) \, , \, f(y) \,) \leq c \times d(x,y)$$

Definition 2.4

A ***Quasi Metric*** is a function $q : U^2 \rightarrow \Re$ such that ,

$$
\begin{array}{lll}
(\mathrm{Q1}) & \forall\, x,y \in U \; . & x = y \;\Leftrightarrow\; q(x,y) = q(y,x) = 0 \\
(\mathrm{Q2}) & \forall\, x,y,z \in U \; . & q(x,z) \leq q(x,y) + q(y,z)
\end{array}
$$

3. Partial Metrics

Definition 3.1

A ***Partial Metric*** [Ma92] is a function $p : U^2 \rightarrow \Re$ such that ,

$$
\begin{array}{lll}
(\mathrm{P1}) & \forall\, x,y \in U \; . & x = y \;\Leftrightarrow\; p(x,x) = p(x,y) = p(y,y) \\
(\mathrm{P2}) & \forall\, x,y \in U \; . & p(x,x) \leq p(x,y) \\
(\mathrm{P3}) & \forall\, x,y \in U \; . & p(x,y) = p(y,x) \\
(\mathrm{P4}) & \forall\, x,y,z \in U \; . & p(x,z) \leq p(x,y) + p(y,z) - p(y,y)
\end{array}
$$

As a metric is precisely a partial metric p such that $\forall\, x \in U \; . \; p(x,x) = 0$ the axioms P1 – P4 specify a class of generalised metrics. P1 – P4 are intended to be the finest possible

generalisation of the metric axioms M1 – M3 such that the distance of a point from itself is not necessarily zero. In [Ma92] it is shown that for each partial metric p the collection ,

$$\{ \ \{ \ y \in U \mid p(x,y) < \varepsilon \ \} \mid x \in U \ \wedge \ \varepsilon > 0 \ \}$$

of ***Open Balls*** is a base for a T_0 partial order quasi metric topology $\mathcal{T}[p]$ of upward closures where the partial ordering $\ll \ \subseteq U^2$ is defined by ,

$$\forall \ x,y \in U \ . \quad x \ll y \quad \Leftrightarrow \quad p(x,x) = p(x,y)$$

Example 3.1
The function $pmax : \Re^2 \rightarrow \Re$ returning the maximum of two non negative real numbers is a partial metric such that ,

$$\forall \ x,y \in \Re \ . \quad x \ll y \quad \Leftrightarrow \quad y \leq x$$

$\mathcal{T}[pmax]$ has the open ball base $\{ \ [0,\varepsilon) \mid \varepsilon > 0 \ \}$.

Example 3.2
The function $int : \{ \ [a,b] \mid a \leq b \ \}^2 \rightarrow \Re$ over the closed intervals on the real line where ,

$$\forall \ a \leq b \ , \ c \leq d \ . \quad int(\ [a,b] \ , \ [c,d] \ ) ::= max\{ b,d \} - min\{ a,c \}$$

is a partial metric such that ,

$$[a,b] \ll [c,d] \quad \Leftrightarrow \quad [c,d] \subseteq [a,b]$$

Example 3.3
For each non empty set S with a special object $\perp \notin$ S the function ,
$p^{\perp} : (\ S \cup \{ \perp \} \ )^2 \rightarrow \{ 0 , 1 \}$ where ,

$$\forall \ x,y \in S \cup \{ \perp \} \ . \quad p^{\perp}(x,y) = 0 \quad \Leftrightarrow \quad x = y \in S$$

defines a *Flat Domain* where ,

$$\forall \ x,y \in S \cup \{ \perp \} \ . \quad x \ll^{\perp} y \quad \Leftrightarrow \quad x = \perp \ \vee \ x = y$$

Example 3.4
For each non empty set S the complete partial order $\langle \ S^* \ , \ \ll^* \ \rangle$ of all finite and infinite sequences over S under the initial segment ordering can be defined by the ***Baire Partial Metric***, $p^* : (S^*)^2 \rightarrow \Re$ where ,

$\forall\ x, y \in S^* \ .\ p^*(x, y) ::= 2^{-n}$ where $n \in \omega \cup \{\infty\}$ is the length of the longest common initial segment between x and y

Example 3.5

For each $n \geq 1$ the finite *n - product* of a partial metric $p : U^2 \rightarrow \Re$ is the partial metric $p^n : (U^n)^2 \rightarrow \Re$ where ,

$$\forall\ x, y \in U^n \ .\ p^n(x, y) ::= \sum_{i \in n} p(x_i, y_i)$$

and , $\forall\ x, y \in U^n \ .\ x \ll^n y \Leftrightarrow \forall\ i \in n \ .\ x_i \ll y_i$

Example 3.6

For each bounded partial metric $p : U^2 \rightarrow \Re$ the *ω - product* of p is the partial metric $p^\omega : (U^\omega)^2 \rightarrow \Re$ where ,

$$\forall\ x, y \in U^\omega \ .\ p^\omega(x, y) ::= \sum_{i \in \omega} p(x_i, y_i) \times 2^{-i}$$

and , $\forall\ x, y \in U^\omega \ .\ x \ll^\omega y \Leftrightarrow \forall\ i \in \omega \ .\ x_i \ll y_i$

The partial metric is the finest generalisation of a metric which allows the distance of an object from itself to be not necessarily zero. The reason for this is based upon the following philosophical understanding of domain theory. Classic incompleteness results force us to add *undecidable* objects such as $\perp$ to the semantic domain for any non trivial programming language. In particular it is not possible to decide if $\perp$ is equal to itself. For example, for any monotonic equality function of the form ,

$$eq : \{true, false, \perp\}^2 \rightarrow \{true, false, \perp\}$$

over a flat domain we cannot have $eq(\perp, \perp) = true$. The conclusion drawn by the author from this is that the metric *decidability axiom* ,

$$(\text{M1}) \qquad \forall\ x, y \in U \ .\ x = y \Leftrightarrow d(x, y) = 0$$

is far too strong to be tenable for a theory of domains as within a metic framework we have to be able to decide that $\perp = \perp$ using $d(\perp, \perp) = 0$.

The philosophy behind the partial metric is that the structure of a Scott style domain can be defined using a distance function which measures the extent to which any two objects can be decided to be equal. For each partial metric $p : U^2 \rightarrow \Re$ and $x, y \in U$, $p(x, y)$ is a numerical measure of the extent to which x & y can be decided equal. The partial metric is a generalisation of the notion of a metric which allows the distance $p(x, x)$ of an object x

from itself to be something other than zero, thus allowing us to attach a notion of ***size*** to each object. This distance is a measure of the degree of ***Completeness*** of x, and so $p(x, x)$ is to be thought of as the *size* of x and is denoted by $|x|$. The following properties which characterise the notion of *size* can be deduced from the axioms P1 – P4.

$$\forall\ x, y \in U \quad . \quad x \ll y \;\Rightarrow\; |x| \geq |y|$$

$$\forall\ x, y \in U \quad . \quad x \ll y \;\wedge\; x \neq y \;\Rightarrow\; |x| > |y|$$

$$\forall\ x, y \in U \quad . \quad x \ll y \;\wedge\; |x| = 0 \;\Rightarrow\; x = y$$

An object x is said to be ***Complete*** if $|x| = 0$ and is said to be ***Partial*** if $|x| > 0$. The subspace of complete objects of a partial metric space is a metric space. The distinction between complete and partial objects is not possible using quasi metrics, and as the cycle contraction mapping theorem below shows, it gives us a powerful tool for reasoning about program correctness. The last of the above properties says that complete objects are always maximal, however, the converse is not always true as the trivial example $p : \{a\}^2 \rightarrow \{1\}$ in which the maximal object a must have size 1 shows.

Definition 3.2
For each partial metric $p : U^2 \rightarrow \Re$, $X \in {}^{\omega}U$ is ***Cauchy*** if,

$$\forall\ \varepsilon > 0 \quad \exists\ k \in \omega \quad \forall\ n, m > k \quad . \quad p(X_n, X_m) < \varepsilon$$

Definition 3.3
A partial metric $p : U^2 \rightarrow \Re$ is ***Complete*** if for each $X \in {}^{\omega}U$ there exists $a \in U$ such that,

$$\exists \lim_{n \rightarrow \infty} p(X_n, a) = 0$$

that is, if every Cauchy sequence converges to a complete object.

Note that Definitions 3.2 & 3.3 are consistent with the analogous Definitions 2.2 & 2.3 for metrics.

The Partial Metric Contraction Mapping Theorem
For each complete partial metric $p : U^2 \rightarrow \Re$ and function $f : U \rightarrow U$ such that,

$$\exists\ 0 \leq c < 1 \quad \forall\ x, y \in U \quad . \quad p(f(x), f(y)) \leq c \times p(x, y)$$

f has a unique fixed point, and this point is complete.

This result reduces to Banach's contraction mapping theorem for complete metric spaces [Su75] when p is a metric. For a proof of this result see [Ma92].

Definition 3.4

A partial metric $p : U^2 \rightarrow \Re$ is ***Continuous*** if

1) $\ll$ is chain complete, and the meet of each countable set exists

2) For each chain $X \in U^\omega$. $| \bigsqcup X | = \lim_{n \to \infty} |X_n|$

3) $\forall X \in U^\omega . \; | \sqcap \{ X_n \mid n \in \omega \} | = sup \{ |X_n| \mid n \in \omega \}$

Note that for each continuous partial metric, $\forall x, y \in U . \; | x \sqcap y | = p(x, y)$.
In [Ma92] it is shown that for each continuous partial metric over a set U the continuous functions in $U \rightarrow U$ are precisely the chain continuous functions.

4. Complete and Partial Objects

In the early days of programming language theory total correctness was defined as partial correctness plus termination [Ho69], an idea now rendered largely obsolete by the need for ever more non terminating software which is intended to be totally correct. The concept of termination does not generalise to such infinite behaviours because termination is by definition something which can occur only after a finite number of steps. One way around this problem is to formulate total correctness for both finite and infinite behaviours as the limit of an infinite sequence of *finite* correctness properties. For example, if $P^\omega : U \rightarrow Boolean$ is a total correctness property for a domain of behaviours U, and if for each $n \geq 0$, $P^n : U \rightarrow Boolean$ is a finite correctness property then the following proof rule can be used to handle infinite behaviours.

$$\forall x \in U . \; (\forall n \geq 0 . \; P^n(x)) \Rightarrow P^\omega(x)$$

The concept of size in a partial metric space gives us a means of expressing finite properties such as,

$$\forall x \in U \;\; \forall n \geq 0 . \;\; P^n(x) \;\; ::= \;\; |x| < 2^{-n}$$

where *finiteness* is formulated as *to within a certain size.* In constructing a framework for reasoning about the total correctness of programs we aim to choose a partial metric for behaviours in which the totally correct ones are precisely the complete objects. In the case of functional programming languages where a behaviour is formulated as an evaluation of a data object the description of completeness given by Wadge [Wa81] is the most appropriate.

" A complete object (in a domain of data objects) is , roughly speaking , one which has no holes or gaps in it , one which cannot be further completed. "

Using Kahn's model of *Data Flow* computation [Ka74] as an example Wadge presented a convincing argument for completeness, only hinting at how it might be possible to generalise this work to other models such as that used by the *Lucid* [W&A85] lazy data flow programming language. The partial metric succeeds in making the *big break* for completeness from the restrictive world of Kahn data flow to other models of computation based upon Scott style topologies.

Kahn's data flow model of computation is a finite asynchronous message passing network of sequential deterministic processes communicating via unidirectional Unix style pipes. The denotational semantics of a network of n processes over a message set S is the least fixed point $Y(F)$ of a chain continuous function $F : (S^*)^n \rightarrow (S^*)^n$. Wadge demonstrated that if there exists a function $M : n^2 \rightarrow \{\ldots, -1, 0, 1, \ldots, \infty\}$ such that ,

$$\forall\, x \in (S^*)^n \;\; \forall\, i \in n \;.\;\; length\ F(x)_i \;\geq\; \min_{j \in n} (\, length\, x_j + M_{ij} \,)$$

and satisfying the ***Cycle Sum Test*** in which all ***Cycle Sums*** of the form ,

$$M_{ab} + M_{bc} + M_{cd} + \ldots + M_{ij} + M_{ja}$$

must be strictly positive then the network will not deadlock , that is ,

$$\forall\, i \in n \;.\;\; length\ Y(F)_i = \infty$$

Neither the statement nor the proof of the cycle sum test can generalise to other domains such as in Example 3.6 where there is no concept of *length* . However, if the notion of length is generalised to be the distance of an object from itself in the context of a generalised metric then considerable progress can be made.

5. The Cycle Contraction Mapping Theorem

In [Ma85] the suggestion made by Wadge [Wa81] ,

> *"It is not possible as far as we know to formulate the cycle sum theorem purely in terms of functions on an abstract metric space."*

was refuted using a formulation of the theorem in terms of a generalised metric $d : U^2 \rightarrow \Re$ satisfying the axioms,

$$\forall\ x, y \in U \quad .\quad d(x, y) = 0 \;\Rightarrow\; x = y$$
$$\forall\ x, y \in U \quad .\quad d(x, y) = d(y, x)$$
$$\forall\ x, y, z \in U \quad .\quad d(x, z) \leq d(x, y) + d(y, z)$$

Although suitable for formulating a version of the cycle sum test for all complete metric spaces this generalised metric does not in general have an open ball topology, and so cannot be used to extend the theorem to a class of Scott style topologies such as those definable by partial metrics. In particular the cycle sum test for Kahn data flow cannot be applied to Lucid programs using this approach. The *Cycle Contraction Mapping Theorem* is both the generalisation of the cycle sum test from the Kahn domain to all complete partial metric spaces and the generalisation of Banach's contraction mapping theorem to all complete partial metric spaces, and is formulated as a result to prove unique fixed points for functions of the form $F : U^n \rightarrow U^n$. The first step is to generalise the notion of Banach's contraction constant $0 \leq c < 1$ to an array of constants.

Definition 5.1
A ***Semi Cycle Contraction Constant*** is a function of the form $c : n^2 \rightarrow \Re$

Definition 5.2
A ***Semi Cycle Contraction*** is a function $F : U^n \rightarrow U^n$ for which there exists a semi cycle contraction constant $c : n^2 \rightarrow \Re$ such that ,

$$\begin{aligned} &\forall\ x, y \in U^n \quad \forall\ i \in n \quad . \quad p(\ F(x)_i\ ,\ F(y)_i\ ) \\ &\qquad \leq \quad max\ \{\ c(i, j) \times p(\ x_j\ ,\ y_j\ ) \mid j \in n\ \} \end{aligned}$$

Lemma 5.1
For each semi cycle contraction $F : U^n \rightarrow U^n$ with semi cycle contraction constant c ,

$$\begin{aligned} &\forall\ m \geq 1 \quad \forall\ x, y \in U^n \quad \forall\ j_0 \in n \quad . \\ &\quad p(\ (F^m(x))(j_0)\ ,\ (F^m(y))(j_0)\ ) \\ &\qquad \leq max\ \{\ c(\ j_0\ ,\ j_1\ ) \times c(\ j_1\ ,\ j_2\ ) \times \ldots \\ &\qquad\qquad \times\ c(\ j_{m-1}\ ,\ j_m\ ) \times p(\ x(j_m)\ ,\ y(j_m)\ ) \\ &\qquad\qquad \mid\ j_1\ ,\ \ldots\ ,\ j_m \in n\ \} \end{aligned}$$

Proof :

Suppose $F : U^n \rightarrow U^n$ is a semi cycle contraction with semi cycle contraction constant c , and that $m \geq 1$.

The proof is by induction on m .

True for $m = 1$ by Definition 5.2 .

By induction suppose true for some $m \geq 1$, then,

$$\forall\ x, y \in U^n \ \forall\ j_0 \in n \ .$$

$$p(\ (F^{m+1}(x))(j_0)\ ,\ (F^{m+1}(y))(j_0)\ )$$

$$\leq max\ \{\ c(\ j_0\ , j_1\ ) \times p(\ (F^m(x))(j_1)\ ,\ (F^m(y))(j_1)\ )\ \mid\ j_1 \in n\ \}$$

$$\leq max\ \{\ c(\ j_0\ , j_1\ ) \times max\ \{\ c(\ j_1\ , j_2\ ) \times \ldots \times c(\ j_{m-1}\ , j_m\ ) \times p(\ x(j_{m+1})\ ,\ y(j_{m+1})\ )\ \mid\ j_2\ , \ldots , j_{m+1} \in n\ \}\ \mid\ j_1 \in n\ \}$$

(by the induction hypothesis)

$$= max\ \{\ c(\ j_0\ , j_1\ ) \times c(\ j_1\ , j_2\ ) \times \ldots \times c(\ j_m\ , j_{m+1}\ ) \times p(\ x(j_{m+1})\ ,\ y(j_{m+1})\ )\ \mid\ j_1\ , \ldots , j_{m+1} \in n\ \}$$

❑

Definition 5.3

For each semi cycle contraction constant $c : n^2 \rightarrow \Re$ and $m \geq 1$ a ***Path*** ρ ***of Length*** $\#\rho \geq 1$ is a function $\rho : \{0, \ldots, \#\rho\} \rightarrow n$. ρ is a ***Cycle*** if $\rho_0 = \rho_{\#\rho}$, and ρ is ***Cycle-free*** if $\forall\ 0 \leq i \neq j \leq \#\rho\ .\ \rho_i \neq \rho_j$. The ***Product*** of ρ is,

$$\rho^* \ ::= \ \times\ \{\ c(\ \rho_i\ , \rho_{i+1}\ )\ \mid\ i \in \#\rho\ \}$$

The sub paths $\langle \rho_i\ , \ldots , \rho_j \rangle$ and $\langle \rho_{i'}\ , \ldots , \rho_{j'} \rangle$ of ρ are ***Disjoint*** if $j \leq i'$ or $j' \leq i$.

Lemma 5.2

Every cycle-free path for a semi cycle contraction constant $c : n^2 \rightarrow \Re$ has length less than n.

Proof :

Suppose ρ is a cycle-free path for a semi cycle contraction constant $c : n^2 \rightarrow \Re$
Then, $\forall\ i \neq j \in \{0, \ldots, \#\rho\}\ .\ \rho_i \neq \rho_j$
Thus the cardinality of $\{\rho_0, \ldots, \rho_{\#\rho}\}$ is $\#\rho + 1$
But, $\{\rho_0, \ldots, \rho_{\#\rho}\} \subseteq n$
Thus, $\#\rho + 1 \leq n$
Thus, $\#\rho < n$

❑

Lemma 5.3
Each path ρ for a semi cycle contraction constant $c : n^2 \rightarrow \Re$ has at least $\lfloor \#\rho / n \rfloor$ disjoint cycles.

Proof :

Suppose ρ is a path for a semi cycle contraction constant $c : n^2 \rightarrow \Re$
Then ρ has the disjoint sub paths ,

$$\langle \rho_0 , \rho_1 , \ldots , \rho_n \rangle$$
$$\langle \rho_n , \rho_{n+1} , \ldots , \rho_{2 \times n} \rangle$$
$$\ldots\ldots\ldots$$
$$\langle \rho_{k-n} , \rho_{k-n+1} , \ldots , \rho_k \rangle$$

where $k ::= \lfloor \#\rho / n \rfloor \times n$

Then by Lemma 5.2 each of these disjoint sub paths has at least one cycle , and so ρ has at least $\lfloor \#\rho / n \rfloor$ disjoint cycles .

❑

Definition 5.4
A ***Cycle Contraction Constant*** is a semi cycle contraction constant pasing the ***Cycle Product Test*** ,

$$\forall \rho \ . \quad \rho_0 = \rho_{\#\rho} \quad \Rightarrow \quad \rho^* < 1$$

which says that the product of every cycle must be less than 1 . To see that the cycle sum test is an instance of the cycle product test suppose that c is such that we can find a unique function $M : n^2 \rightarrow \{ \ldots , -1 , 0 , 1 , \ldots , \infty \}$ such that ,

$$\forall \ i , j \in n \ . \quad c(i , j) = 2^{-M(i , j)}$$

Then the cycle product test is equivalent to the cycle sum test,

$$\forall \ \rho \ . \quad \rho_0 = \rho_{\#\rho} \quad \Rightarrow \quad \sum_{i \in \#\rho} M(\rho_i , \rho_{i+1}) > 0$$

Lemma 5.4
For each cycle contraction constant ,

$$sup \{ \ \rho^* \mid \rho_0 = \rho_{\#\rho} \ \} < 1$$

Proof :

Suppose $c : n^2 \rightarrow \mathfrak{R}$ is a cycle contraction constant

Suppose ρ is a cycle .

Then by Lemma 5.2 we can keep removing sub cycles of ρ to find a sub cycle ρ' of ρ such that $\#\rho' < n$.

Also $\rho^* \leq \rho'^*$ as c passes the cycle product test .

$$
\begin{aligned}
\text{Thus ,} \quad & sup \{ \ \rho^* \mid \rho_0 = \rho_{\#\rho} \ \} \\
& \leq sup \{ \ \rho^* \mid \rho_0 = \rho_{\#\rho} \wedge \#\rho < n \ \} \\
& < 1 \\
& \qquad \text{as } \{ \ \rho \mid \rho_0 = \rho_{\#\rho} \wedge \#\rho < n \ \} \text{ is finite}
\end{aligned}
$$

❑

Lemma 5.5

For each cycle contraction constant $c : n^2 \rightarrow \mathfrak{R}$,

$$\exists \ m \geq 1 \quad \forall \ \rho \ . \quad \#\rho = m \quad \Rightarrow \quad \rho^* \leq 1 / (2 \times n)$$

Proof :

Suppose $c : \{ 1 , \ldots , n \}^2 \rightarrow \mathfrak{R}$ is a cycle contraction constant

Thus by Lemmas 5.2 & 5.3 ,

$$
\begin{aligned}
& \forall \ \rho \quad \exists \ k \geq \lfloor \#\rho / n \rfloor \quad . \quad \rho^* \leq a \times b^k \\
& \quad \text{where ,} \quad a ::= sup \{ \ \rho'^* \mid \forall \ 0 \leq i \neq j \leq \#\rho' \ . \ \rho'_i \neq \rho'_j \ \} \\
& \qquad\qquad\quad b ::= sup \{ \ \rho' \mid \rho'_0 = \rho'_{\#\rho'} \ \}
\end{aligned}
$$

But by Lemma 5.4 $b < 1$, thus for large enough $\#\rho$ the result follows.

❑

Lemma 5.6

For each function $F : U^n \rightarrow U^n$ and $m \geq 1$, if F^m has a unique fixed point then this point is also the unique fixed point of F .

Proof :

Suppose $F^m : U^n \rightarrow U^n$ has the unique fixed point $a \in U^n$, and $m \geq 1$

$$
\begin{aligned}
\text{Then ,} \quad & a = F^m (a) \\
\therefore \quad & F(a) = F(\ F^m (a) \ ) \\
\therefore \quad & F(a) = F^m (\ F(a) \ ) \\
\therefore \quad & a = F(a) \quad \text{as } a \text{ is unique}
\end{aligned}
$$

And so F is shown to have a as a fixed point ; now we show it to be unique

Suppose $b \in U^n$ is such that

$$b = F(b)$$

$$\therefore \quad F(b) = F^2(b)$$

$$\therefore \quad F^2(b) = F^3(b)$$

$$\ldots\ldots\ldots\ldots$$

$$\therefore \quad F^{m-1}(b) = F^m(b)$$

Thus , $b = F^m(b)$

Thus , $a = b$ as a is unique.

❑

Definition 5.5

A ***Cycle Contraction*** is a semi cycle contraction having a cycle contraction constant. Clearly a Banach contraction mapping is precisely a cycle contraction mapping where p is a metric and $n = 1$.

The Cycle Contraction Mapping Theorem

A cycle contraction over a complete partial metric space has a unique fixed point , and this point is complete.

Proof :

Suppose $F : U^n \rightarrow U^n$ is a cycle contraction with cycle contraction constant c .

By Lemma 5.6 and the *Partial Metric Contraction Mapping Theorem* it is sufficient to show that there exists $m \geq 1$ such that F^m is a contraction

By Lemma 5.5 we can choose $m \geq 1$ such that ,

$$\forall \rho \ . \ \#\rho = m \ \Rightarrow \ \rho^* \leq 1 / (2 \times n)$$

Thus using Lemma 5.1 for all $x , y \in U^n$,

$$p^n(F^m(x) , F^m(y))$$

$$= \Sigma \{ \ p(F^m(x)_i , F^m(y)_i) \mid i \in n \ \}$$

$$\leq \Sigma \{ \ max \{ \ p(x(j_m) , y(j_m)) / (2 \times n) \mid j_1 , \ldots , j_m \in n \ \} \mid i \in n \ \}$$

$$= \Sigma \ \{ \ max \{ \ p(x_j , y_j) \mid j \in n \ \} \mid i \in n \ \} / (2 \times n)$$

$$= 1/2 \times max\{ p(x_j, y_j) \mid j \in n \}$$

$$\leq 1/2 \times p^n(x, y)$$

❑

6. The Complete Cycle Contraction Mapping Theorem

Proof theory for programs, such as in safety critical systems, often requires a correctness proof for a program which is intuitively obviously correct. In our case this means proving properties over partial metric spaces of recursively defined functions which have unique & complete fixed points. Thus if no partial objects are involved in such definitions it should not be necessary to have to use partial objects in a correctness proof. A simple example from Lucid is the definition,

$$x = fby(\ one\ ,\ plus(\ x\ ,\ one\ )\ )$$

where,

$$\forall\ i \in \omega\ .\ \ one_i\ ::=\ 1$$

$$\forall\ x, y\ \ \forall\ i \in \omega\ .\ \ plus(x, y)_i\ ::=\ x_i + y_i$$

$$\forall\ x, y\ \ \forall\ i \in \omega\ .\ \ fby(x, y)_i\ \begin{array}{ll} ::= x_0 & \text{if } i = 0 \\ ::= x_{i-1} & \text{if } i > 0 \end{array}$$

It is no secret that the function $\lambda\ x\ .\ fby(\ one\ ,\ plus(\ x\ ,\ one\ )\ )$ has the unique & complete fixed point $\lambda\ i \in \omega\ .\ i+1$, but is there any way of proving such *obvious* results without reference to either partial objects or approximations ? This was a question posed in [Wa81] for which we can use the cycle contraction mapping theorem to give a positive answer.

Definition 6.1
For each partial metric $p : U^2 \rightarrow \Re$, a function $f : U \rightarrow U$ is ***Optimal*** if,

$$\forall\ x, y \in U\ .\ \ x \ll y \ \Rightarrow\ f(x) \ll f(y) \qquad \text{and,}$$

$$\forall\ x, y \in U\ \ \exists\ x', y' \in U\ .$$
$$x \ll x' \ \wedge\ y \ll y' \ \wedge\ |x'| = |y'| = 0 \ \wedge$$
$$p(\ f(x)\ ,\ f(y)\ ) \ =\ p(\ f(x')\ ,\ f(y')\ )$$

Lemma 6.1

For each partial metric $p : U^2 \rightarrow \Re$,

$$\forall\ x \ll x',\ y \ll y' \in U^n\ .\ \ p^n(x, y) = p^n(x', y') \\ \Leftrightarrow\ \forall\ i \in n\ .\ p(x_i, y_i) = p(x'_i, y'_i)$$

The Optimal Cycle Contraction Mapping Theorem

If the restriction $F \mid \{ x \in U^n \mid |x| = 0 \}$ to the complete objects of an optimal function $F : U^n \rightarrow U^n$ over a complete partial metric $p : U^2 \rightarrow \Re$ is a cycle contraction then F has a unique fixed point and this point is complete.

Proof :

Suppose $p : U^2 \rightarrow \Re$ is a complete partial metric

Suppose $F : U^n \rightarrow U^n$ is an optimal function such that $F \mid \{ x \in U^n \mid |x| = 0 \}$ is a cycle contraction with cycle contraction constant c .

Suppose $x, y \in U^n$

Then as F is optimal we can choose $x', y' \in U^n$ such that,

$$x \ll x' \ \wedge\ y \ll y' \ \wedge\ |x'| = |y'| = 0 \ \wedge \\ p^n(F(x), F(y)) = p^n(F(x'), F(y'))$$

Thus , $\forall\ i \in n\ .\ p(F(x)_i , F(y)_i)$

$= p(F(x')_i , F(y')_i)$ (by Lemma 6.1 and as F is monotonic)

$\leq max \{ c(i,j) \times p(x'_j , y'_j) \mid j \in n \}$

$\leq max \{ c(i,j) \times p(x_j , y_j) \mid j \in n \}$

(as $\forall\ j \in n\ .\ p(x'_j , y'_j) \leq p(x_j , y_j)$)

Thus the theorem follows by the cycle contraction mapping theorem

❑

7. Conclusions and Further Work

The principle conclusion from the work in this report is that a theory of complete & partial objects as envisaged by Wadge is possible by using partial metrics to generalise the structure of a complete metric space to include partial objects. The cycle contraction mapping theorem supports this conclusion as it generalises both the cycle sum test and Banach's theorem. This work is still a long way from the desired goal of finding suitable partial metrics for function spaces, and from

there to a reflexive theory of domains. If possible we would effectively have Scott's pioneering ideas on denotational semantics [St77] combined with a notion of completeness. This work is a convincing argument that a generalised metric approach to denotational semantics is plausible, an argument which does not fall foul of the traditional prejudice held against quasi metrics that they are not symmetric. After all, according to legend, it was Scott himself who said that domains should be metrizable.

The next task is to demonstrate that obviously totally correct programs can be reasoned about without reference to partial objects and approximation. The challenge is to design a programming language of optimal functions to which the optimal cycle contraction mapping theorem can be applied. The author is now attempting this by constructing algebras of optimal functions for Landin's sugared λ - calculus ISWIM [La64] notation.

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Addendum

RR228 The Cycle Contraction Mapping Theorem
Steve Matthews

The transitivity axiom *(P4)* used in the definition of a *Partial Metric* was first suggested in 1987 by Steve Vickers of Imperial College in notes entitled "Matthews Metrics". These notes were an exploration and development of ideas in my Ph.D. thesis [Ma85]. These notes were kindly given to me in 1987 by Steve, and then sadly forgotten until now. Unfortunately I both believed, and so went around telling everybody, that I introduced *(P4)*, while in fact it first appeared in Steve's notes. Sorry Steve ! Working in forgetful ignorance of Steve's notes the axiom *(P4)* later reappeared independently (if you see what I mean !) in my later work on the *Partial Metric*. This experience of forgetfulness has produced an interesting result though. The reflexive axiom *(P1)* which, I currently believe I first introduced, appears to be "hinted" at in Steve's notes although not actually fomulated. This means that the Partial Metric appears to be the required formulation for ideas discussed by Steve in his notes.

Steve Matthews Feb. 17th 1993